Quantum Operations continued

A lecture series by IQM and HS RM

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In this session, we will dive deeper into the single- and two-qubit gates we have discovered last time

Agenda

1. A Brief Introduction to Quantum Mechanics

- A short introduction to quantum mechanics is given to make this training self-contained.

2. One-Qubit Gates

- Quantum gates are quantum counterparts of classical gates such as NOT, OR and AND.
- Here simple quantum gates acting on 1-qubit are introduced. Tensor product is mathematically defined.

3. Two-Qubit Gates

- Two-qubit gates have two inputs and two outputs. They are necessary to entangle tensor product states.
- Any unitary gate acting on an n-qubit state (a $2^n \times 2^n$ big matrix) can be decomposed into 1-qubit gates and 2-qubit gates.

A Brief Introduction to Quantum Mechanics

- Information can be processed with complex vectors & linear operators (matrices).
- Quantum mechanics describes the microscopic world, where states are represented by complex vectors and manipulations (gates) of the states are represented by matrices.
- Rule 1: A quantum state is described by a vector $|\psi\rangle \in \mathbb{C}^n$, where n is a positive integer. $|\psi\rangle$ is normalized and a linear combination (superposition) of two vectors is another vector $c_1|\psi_1\rangle + c_2|\psi_2\rangle \in \mathbb{C}^n$.
- Rule 2: For any physical quantity a (observable), there is a corresponding Hermitian matrix A ($A^* = A$). Upon measurement of a, the outcome is one of the real numbers (eigenvalue) associated with A and the state right after the measurement is a special vector (eigenvector) corresponding to the eigenvalue.

1. A Brief Introduction to Quantum Mechanics

- Rule 3: Among observables, energy is most important and deserves a special name, the Hamiltonian. It tells the state $|\psi\rangle$ how to evolve in time:

$$i\hbar \frac{d}{dt}|\psi\rangle = H|\psi\rangle$$ (the Schrödinger equation)

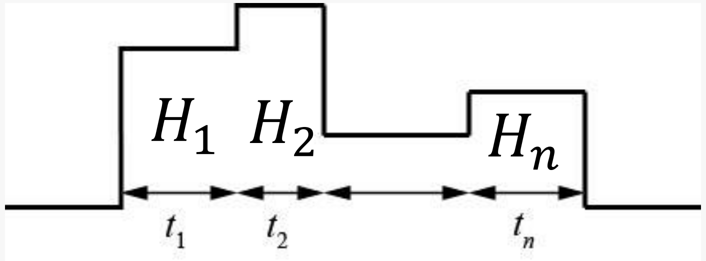
where $\hbar$ is a constant known as the Planck constant. Often put $\hbar = 1$.

Remarks:

- $e^{i\alpha}|\psi\rangle \sim |\psi\rangle$ .There is no way to measure the phase.
- The Schrödinger equation is formally solved as

  $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$ when H is independent of time.
- If the Hamiltonian is H_1 for the 1st duration t_1 ,H_2 for 2nd duration t_2 ,… ,H_N for the N^{th} duration t_N, then

  $|\psi(t)\rangle = e^{-iH_N t_N} \ldots e^{-iH_2 t_2} e^{-iH_1 t_1}|\psi(0)\rangle$.
- Each $e^{-iH_k t_k}$ roughly corresponds to a gate in quantum computing.



Exercise

- Let $|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $H_1 = a\,\sigma_x$,$H_2 = b\,\sigma_z$,where $\sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Suppose H_1 acts first for a time-duration $t_1 = \frac{\pi}{2}$ then H_2 acts for a time-duration $t_2 = \frac{\pi}{4}$. Find the state $|\psi(t)\rangle$ at time $t = t_1 + t_2$. You may put $\hbar = 1$ and use

$$e^{i\,\alpha\,\sigma_k} = \cos\alpha\, I_2 + i\sin\alpha\,\sigma_k \ .$$

Solution: $e^{-\frac{i\pi}{2}\sigma_x}|\psi(0)\rangle = \sigma_x|\psi(0)\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$e^{-\frac{i\pi}{4}\sigma_z}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{-\frac{i\pi}{4}} & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{\frac{i\pi}{4}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \simeq \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Note: $e^{i\,\alpha\,\sigma_k} = \cos\alpha\, I_2 + i\sin\alpha\,\sigma_k$ is unitary. $e^{i\,\alpha\,\sigma_k}e^{-i\,\alpha\,\sigma_k} = I$. This is true for any gate.

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Qubit Gates and Quantum Circuit

- Any quantum gate/circuit U maps a quantum state $|\psi_{in}\rangle$ to a quantum state $|\psi_{out}\rangle = U|\psi_{in}\rangle$.
- It must be unitary since $\langle\psi_{out}|\psi_{out}\rangle = \langle\psi_{in}|U^*U|\psi_{in}\rangle = 1 \rightarrow U^*U = I$.
- Immediate implication: Quantum computing must be reversible. There exists $U^{-1} = U^*$.
- Reversible processes do not produce heat. (Entropy is conserved.)
- Typical HPC costs 10~30 MW power.
- IQM 50QB QC costs 30 kW power, mostly from refrigerator and electronics.
- Number of output qubits must be the same as the number of input qubits.

Quantum Parallelism

- Let $f: \{0,1\}^n \to \{0,1\}^m$, $|\Psi\rangle = \sum_{x=0}^{2^n-1} c_x |x\rangle$ and U_f be a linear operator representing f as $U_f|x\rangle|0\rangle = |x\rangle|f(x)\rangle$. Then U_f acts on $|\Psi\rangle|0\rangle$ as $U_f|\Psi\rangle|0\rangle = \sum_{x=0}^{2^n-1} c_x U_f|x\rangle|0\rangle = \sum_{x=0}^{2^n-1} c_x|x\rangle|f(x)\rangle$. Question: Is $U_f|x\rangle = |f(x)\rangle$ acceptable?

  Note that $|f(x)\rangle$ has been evaluated for 2^n different x simultaneously (quantum parallelism).

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$$
\begin{aligned}
&|0\rangle|0\rangle \mapsto |0\rangle|f(0)\rangle \\
&|1\rangle|0\rangle \mapsto |1\rangle|f(1)\rangle \\
U_f:\ &|2\rangle|0\rangle \mapsto |2\rangle|f(2)\rangle \\
&\quad\vdots \quad \mapsto \quad \vdots \\
&|2^n-1\rangle|0\rangle \mapsto |2^n-1\rangle|f(2^n-1)\rangle
\end{aligned}
$$

by a single action of $U_f \sim 2^n$-core CPU.

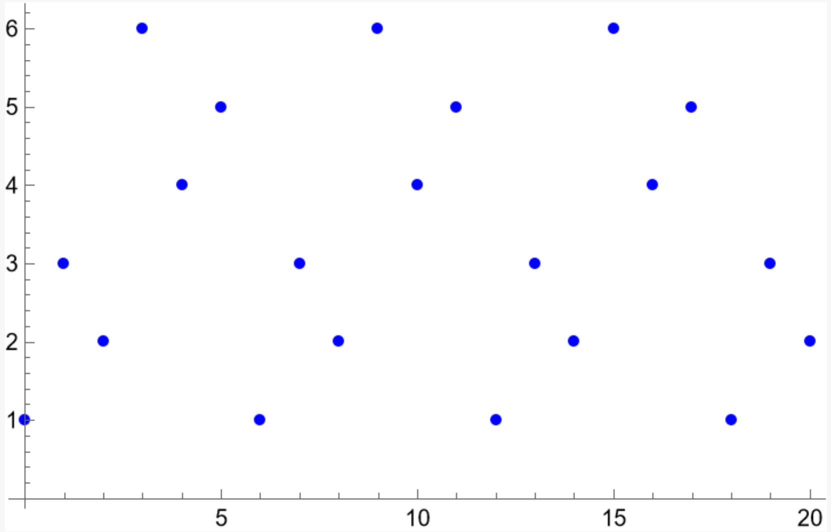
Exercise

- Let $f(x) = 3^x \bmod 7$ and $U_f : |x\rangle|0\rangle \mapsto |x\rangle|f(x)\rangle$ .Let $|\Psi\rangle = \frac{1}{\sqrt{8}}\sum_{x=0}^{7}|x\rangle$. Write down $U_f|\Psi\rangle|0\rangle$ explicitly.

- Solution

$$U_f|\Psi\rangle|0\rangle = \frac{1}{\sqrt{8}}(|0\rangle|1\rangle + |1\rangle|3\rangle + |2\rangle|2\rangle + |3\rangle|6\rangle + |4\rangle|4\rangle + |5\rangle|5\rangle + |6\rangle|1\rangle + |7\rangle|3\rangle)$$

$f(x) = 3^x \bmod 7$ is a periodic function of period 6.



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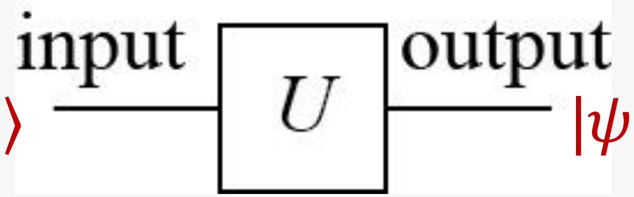
One-Qubit Gates

- A quantum gate that acts on a single qubit is called a 1-qubit gate.

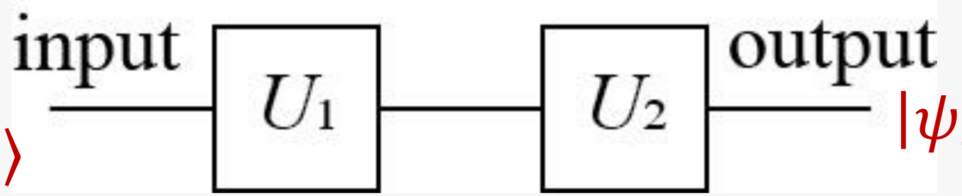
$|\psi_{\text{in}}\rangle$



$|\psi_{\text{out}}\rangle = U|\psi_{\text{in}}\rangle$

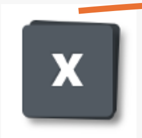
The line denotes a qubit. Time flies from left to right.

$|\psi_{\text{in}}\rangle$



$|\psi_{\text{out}}\rangle = U_2 U_1 |\psi_{\text{in}}\rangle$

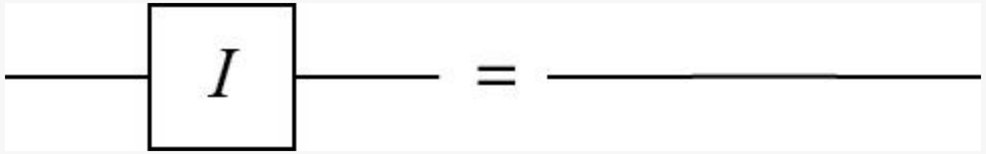
- Recall $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$
- $X = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$. $X|0\rangle = |1\rangle$ & $X|1\rangle = |0\rangle$. $X \leftrightarrow$ NOT.
- $Z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$; $Z|0\rangle = |0\rangle$ & $Z|1\rangle = -|1\rangle$. No classical counterpart.
- $Y = \begin{pmatrix}0 & i\\-i & 0\end{pmatrix}$. $Y = -i\,ZX$. No classical counterpart.



This is how it looked last time

- Hadamard gate $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. It maps $|0\rangle$ and $|1\rangle$ to superposition states.

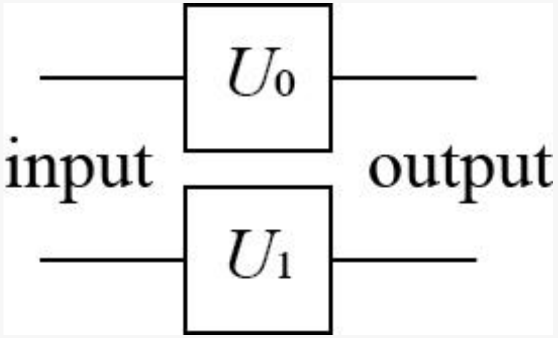
  $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.
- Identity gate $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ does nothing.



- We write $U_1 \otimes U_0$ when U_1 acts on the first qubit and U_0 on the second qubit.

q_0

$|\Psi\rangle$



$(U_1 \otimes U_0)|\Psi\rangle$

q_1

- q_1 and q_0 denote the first qubit and the second qubit and not the qubit states. $|\Psi\rangle$ is entangled in general.

  If $|\Psi\rangle = |\psi_1\rangle|\psi_0\rangle$, $(U_1 \otimes U_0)|\psi_1\rangle|\psi_0\rangle = (U_1|\psi_1\rangle) \otimes (U_0|\psi_0\rangle)$.

Tensor Product

- Tensor product of matrices.
- Let $A = (a_{ij})$ be an $m \times n$ matrix and let $B = (b_{ij})$ be a $p \times q$ matrix. Then $A \otimes B$ is an $mp \times nq$ matrix defined as $A \otimes B = (a_{ij}B) = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}$.
- Example: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 1 \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ 1 \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$
- $\begin{pmatrix} 1 \\ 3 \end{pmatrix} \otimes \begin{pmatrix} 2 \\ i \end{pmatrix} = \begin{pmatrix} 1 \times \begin{pmatrix} 2 \\ i \end{pmatrix} \\ 3 \times \begin{pmatrix} 2 \\ i \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 2 \\ i \\ 6 \\ 3i \end{pmatrix}$.

- $\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \cdot \left[\begin{pmatrix} 1 \\ 3 \end{pmatrix} \otimes \begin{pmatrix} 2 \\ i \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ i \\ 6 \\ 3i \end{pmatrix} = \begin{pmatrix} 6 \\ -3i \\ 2 \\ -i \end{pmatrix}$.
- Note that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$,$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ i \end{pmatrix} = \begin{pmatrix} 2 \\ -i \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 2 \\ -i \end{pmatrix} = \begin{pmatrix} 6 \\ -3i \\ 2 \\ -i \end{pmatrix}$.
- $(A \otimes B) \cdot (|u\rangle \otimes |v\rangle) = (A|u\rangle) \otimes (B|v\rangle)$.
- $(A \otimes B) \cdot (C \otimes D) = (A \cdot C) \otimes (B \cdot D)$ in general whenever the matrix product is well defined.
- Exercise: Evaluate $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$.

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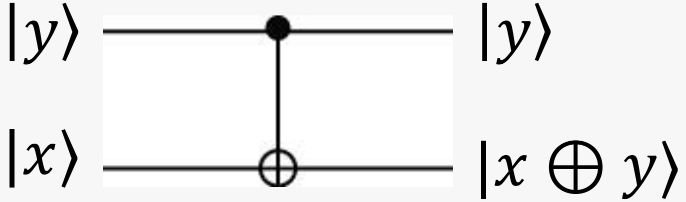
Two-Qubit Gates

- Classical logic circuits require two-bit gates such as NAND and XOR for universality.
- This is also true for quantum circuits. Two-qubit gates are required for universal quantum computation.
- Examples



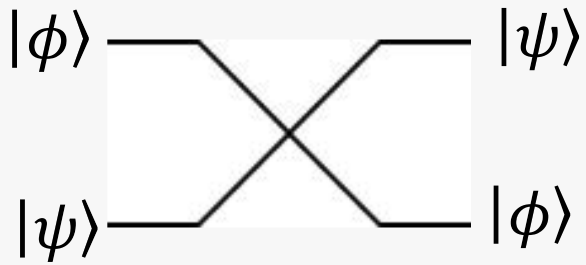
$x \oplus y = x + y \pmod 2$

$0 \oplus 0 = 0$
$0 \oplus 1 = 1$
$1 \oplus 0 = 1$
$1 \oplus 1 = 0$

- $U_{\mathrm{CNOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

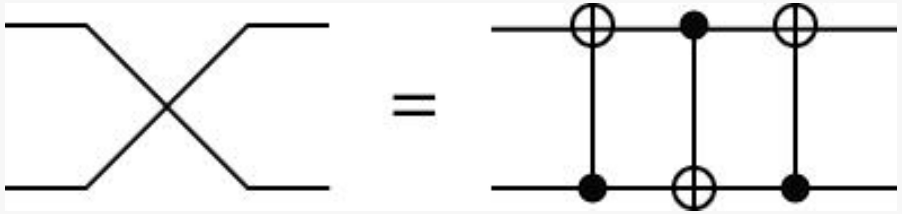
$U_{\mathrm{CNOT}}|x\rangle|y\rangle = |x \oplus y\rangle|y\rangle$. U_{CNOT} flips $|x\rangle$ if $|y\rangle = |1\rangle$. Does nothing to $|x\rangle$ if $|y\rangle = |0\rangle$ .Similar to XOR.

- $U_{\mathrm{SWAP}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $.U_{\mathrm{SWAP}}\,|\psi\rangle|\phi\rangle = |\phi\rangle|\psi\rangle$.

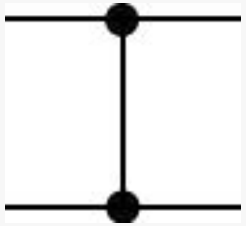


Exercise

- Verify that U_{CNOT} maps $|00\rangle \mapsto |00\rangle, |01\rangle \mapsto |11\rangle, |10\rangle \mapsto |10\rangle, |11\rangle \mapsto |01\rangle$.
- Verify that U_{SWAP} maps $|00\rangle \mapsto |00\rangle, |01\rangle \mapsto |10\rangle, |10\rangle \mapsto |01\rangle, |11\rangle \mapsto |11\rangle$.
- Let $|\Psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$ .Find $U_{\text{CNOT}}|\Psi\rangle$ and $U_{\text{SWAP}}|\Psi\rangle$.
- Verify that $U_{\text{CNOT}} = I_2 \otimes P_0 + X \otimes P_1$,where $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ are called the projection operators.
- Show that



- Let $U_{\text{CZ}} = I_2 \otimes P_0 + Z \otimes P_1$ .Show that $U_{\text{CZ}} = (H \otimes I_2)U_{\text{CNOT}}(H \otimes I_2)$,where H is the Hadamard gate. The symbol of U_{CZ} is



- Note that $I_2 \otimes P_0 + Z \otimes P_1$
  $= P_0 \otimes I_2 + P_1 \otimes Z$.

U_{CZ} is a native quantum gate of an IQM quantum computer.

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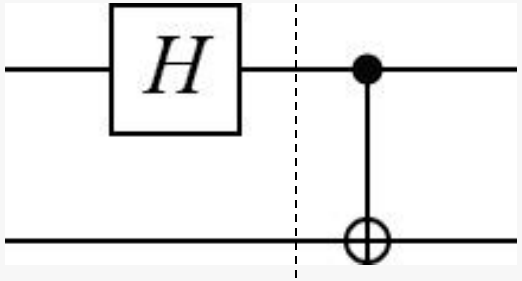
Two-Qubit Gates

- Universality Theorem: Any n-qubit unitary ($\in U(2^n)$) gate can be decomposed into one-qubit gates and CNOT gates.

  The proof is highly technical. See Nielsen & Chuang or Nakahara & Ohmi.
- CNOT above may be replaced by most 2-qubit gates. The SWAP gate and tensor products of 1-qubit gates are exceptions.
- Two-qubit gates are necessary to entangle tensor product states.
- 

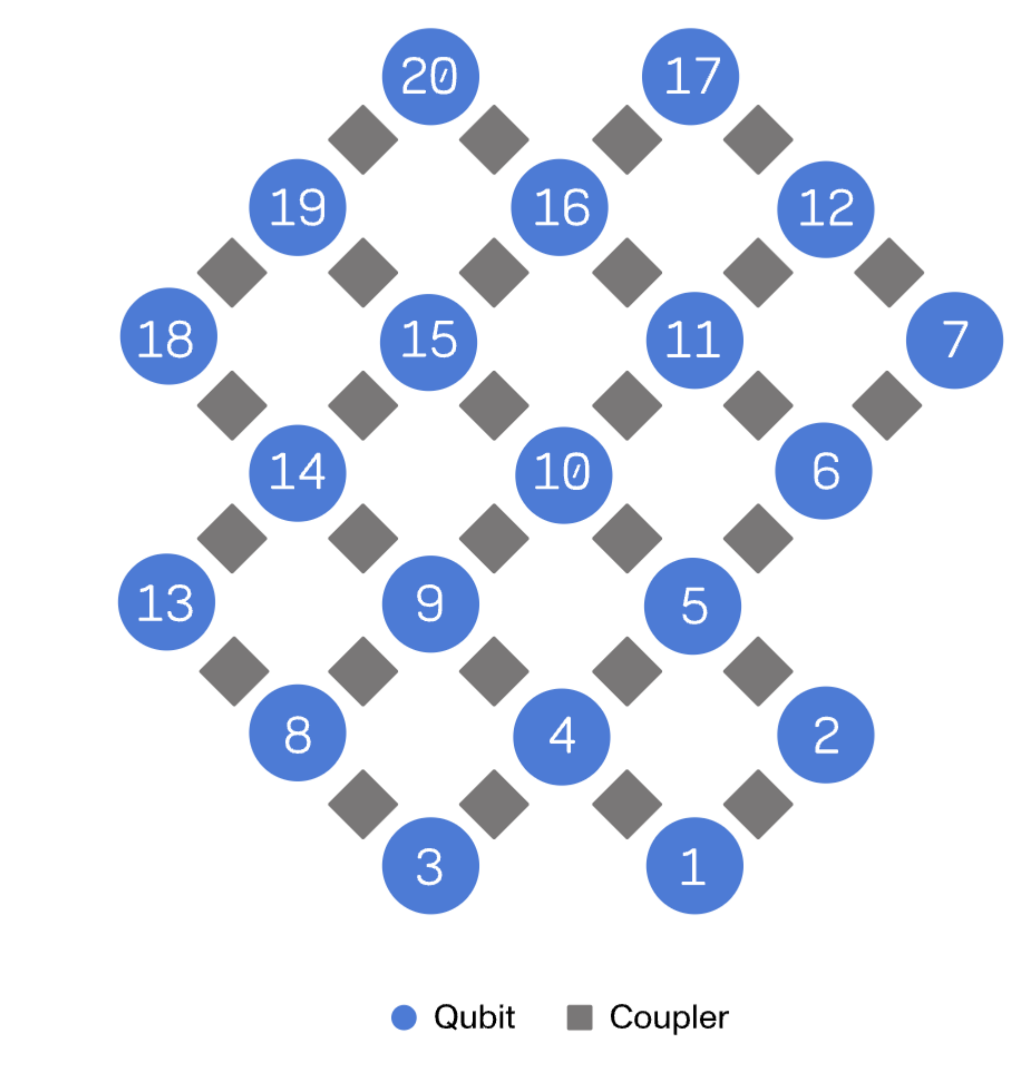
  $|0\rangle$, $|0\rangle$ → $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$

  $\frac{1}{\sqrt{2}}|0\rangle \otimes (|0\rangle + |1\rangle)$

Exercise: $|\Phi_\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $|\Psi_\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$ are called the Bell states. Find quantum circuits that generate $|\Phi_-\rangle$ and $|\Psi_\pm\rangle$ from $|00\rangle$.

Summary

- Any quantum circuit can be decomposed into one-qubit gates and two-qubit gates. The set of one-qubit gates and most two-qubit gates needs to be universal.
- A universal gate set for IQM is CZ and PRX (phased X rotation)
- Decomposition of a given n-qubit circuit into 1- and 2-qubit gates are far from unique. Optimizing the number of gates is an important issue for efficient implementation of quantum algorithms. Optimization depends on QPU topology.



QPU topology of IQM Garnet available via IQM Resonance

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