IQM

Introduction to Quantum States and Quantum Operations

A lecture series by IQM and HS RM

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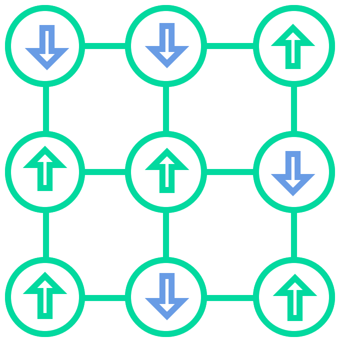
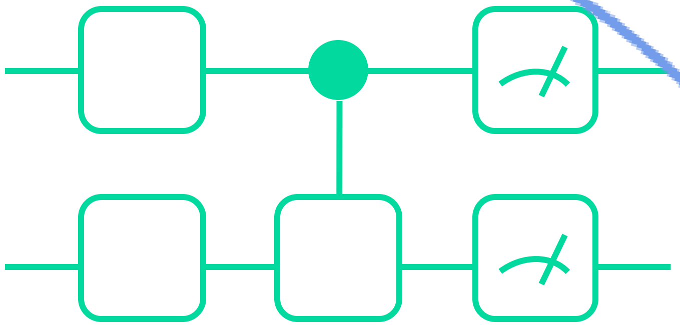
How to work with qubits?

Different ways to work with qubits



Quantum annealing

designed for particular optimization problems



Gate-based quantum computing

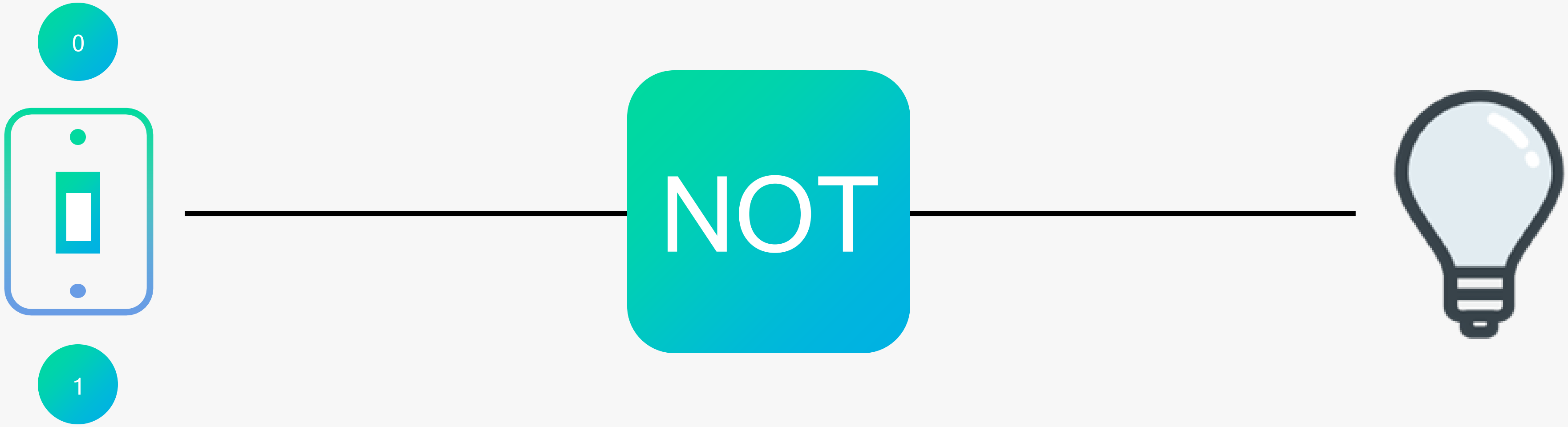
designed for universal (quantum) computing

Conventional computer: bits

- Bits are the foundation of conventional computers

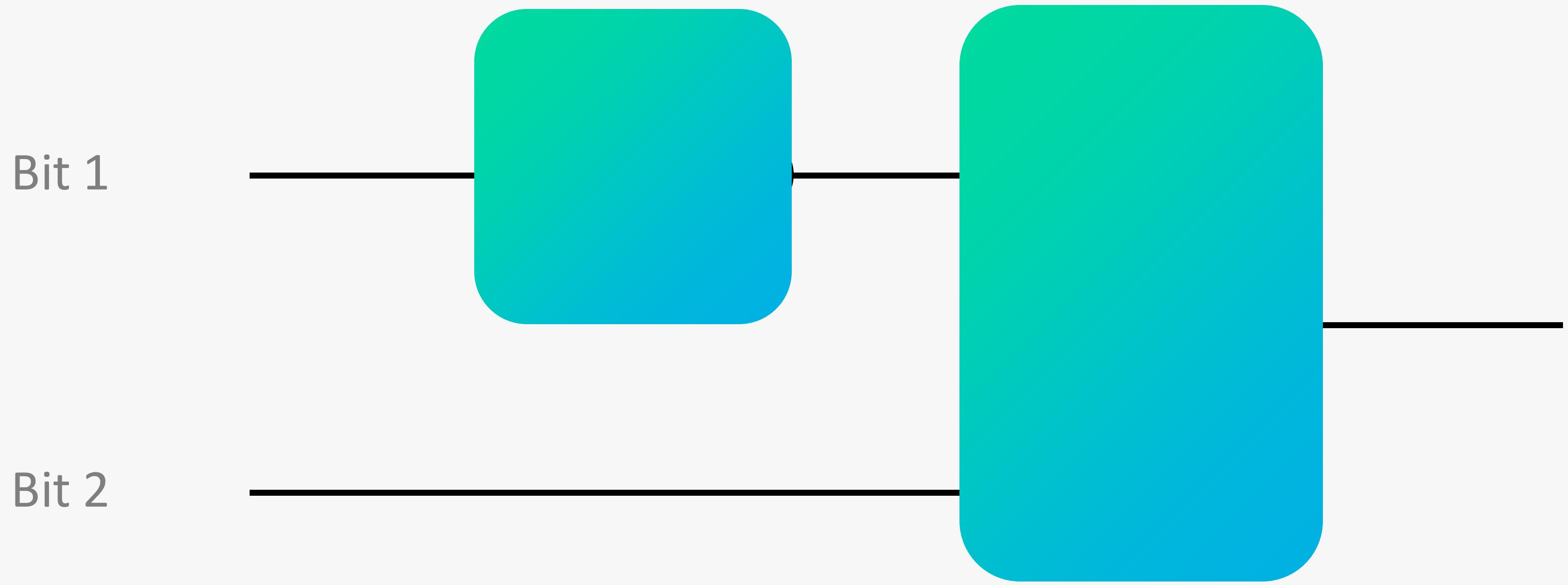
Conventional computer: working with bits

- Information processing is performed with the help of logical gates

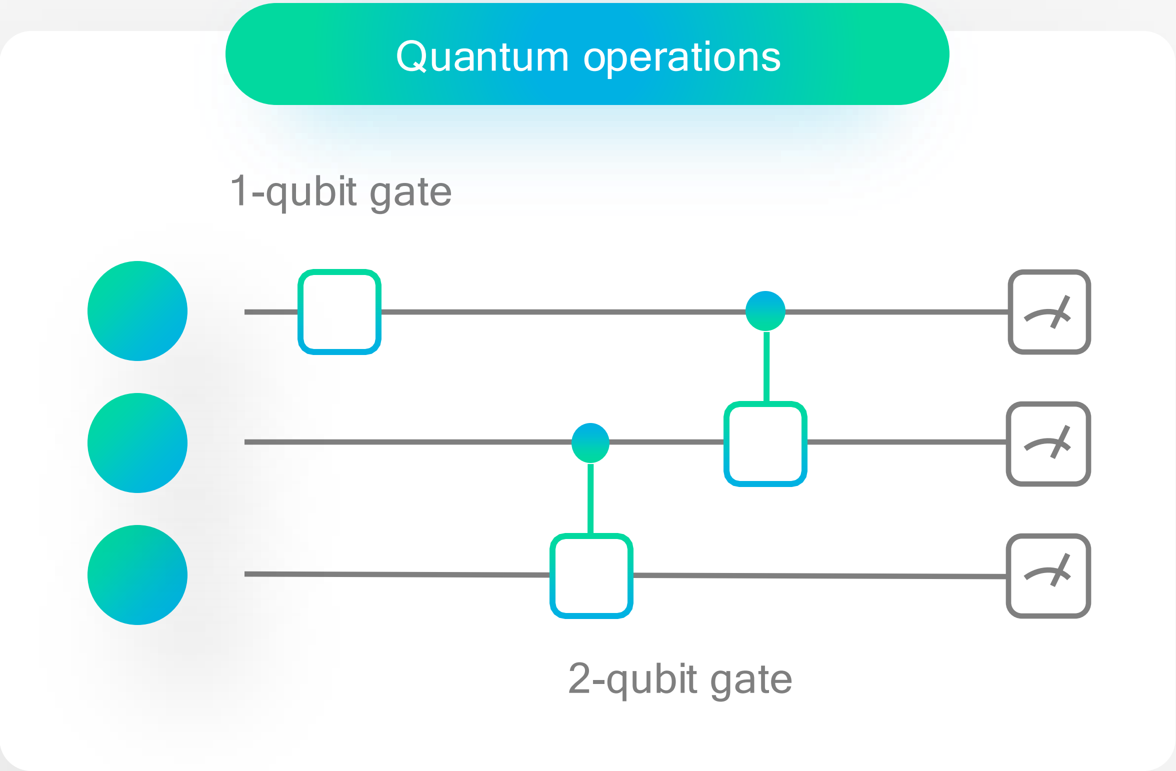


Conventional computer: algorithms

- Multiple gates form a circuit
- An algorithm is implemented in a sequence of gates
- Circuits are represented as a time sequence like musical notes



Quantum computing: algorithms



Activity 1: Exploring the H gate

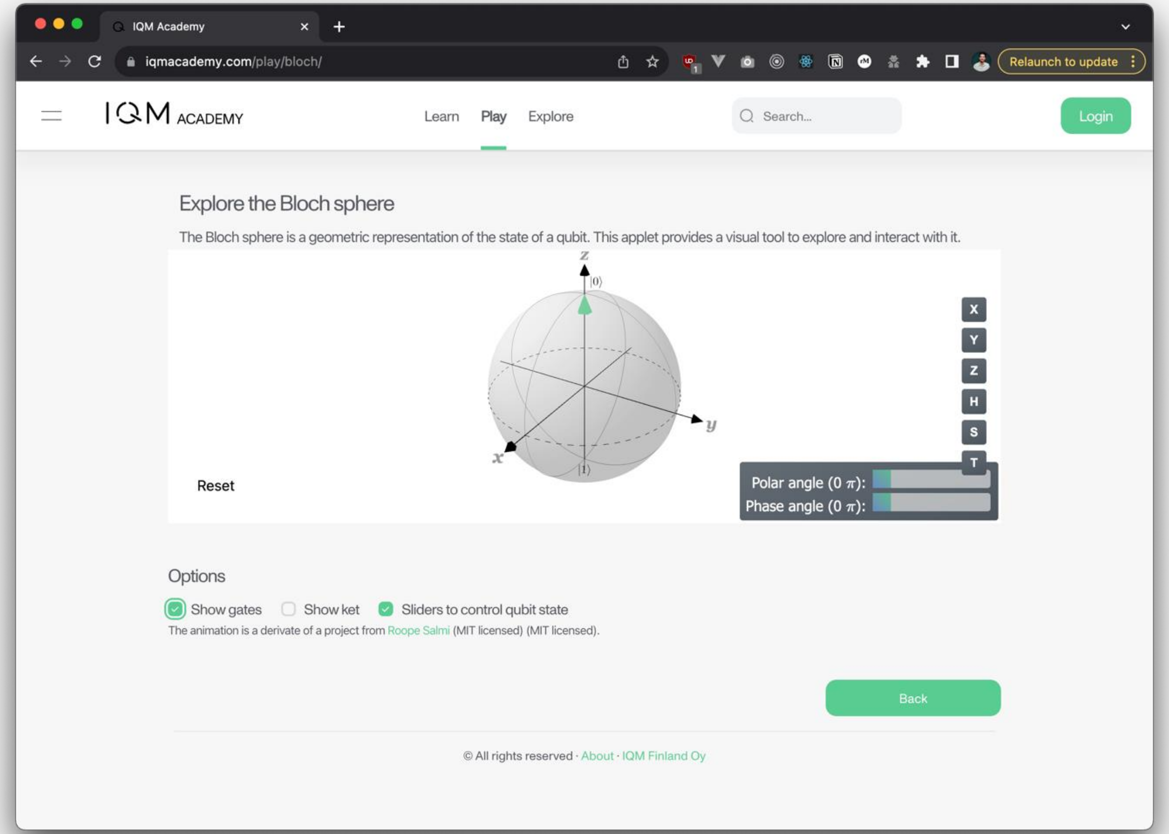
Open https://bit.ly/iqm-1

and investigate the effects of gate H and gate X!

Take it further:

- Apply the Hadamard gate H two times in a row while choosing 0 and 1 as starting values. Describe your observation! What does it mean for the information stored in the qubit?

Activity 1: Bloch Sphere



Open https://www.iqmacademy.com/play/bloch/ and investigate the effects of gate H and gate X again! See what happens if you apply the H gate twice.

Exercise

- Find the Bloch vectors with angles

  (1) $\theta = \pi, \phi = 0$

  (2) $\theta = \pi/2, \phi = 0$

  (3) $\theta = \frac{\pi}{2}, \phi = -\pi/2$

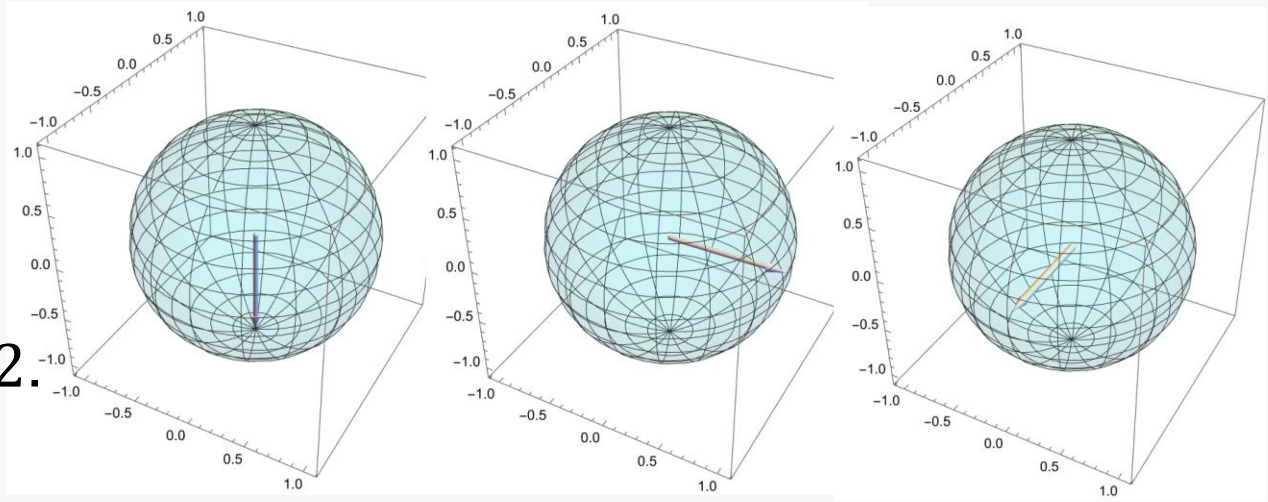
- Find the complex vectors corresponding to these Bloch vectors.

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

(1) $\theta = \pi, \phi = 0$: $|\psi\rangle = |1\rangle$.

(2) $\theta = \pi/2, \phi = 0$: $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$.

(3) $\theta = \frac{\pi}{2}, \phi = -\pi/2$: $|\psi\rangle = (|0\rangle - i|1\rangle)/\sqrt{2}$.



Activity 1: Description of a quantum state

A single-qubit quantum state $|\psi\rangle$ can be described mathematically as

$$|\psi\rangle = a|0\rangle + b|1\rangle \text{ with } |a|^2 + |b|^2 = 1$$

with $\mathrm{a}, \mathrm{b} \in \mathbb{C}$

This is called a ket,
Dirac notation for
vectors

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Activity 1: Description of a quantum state

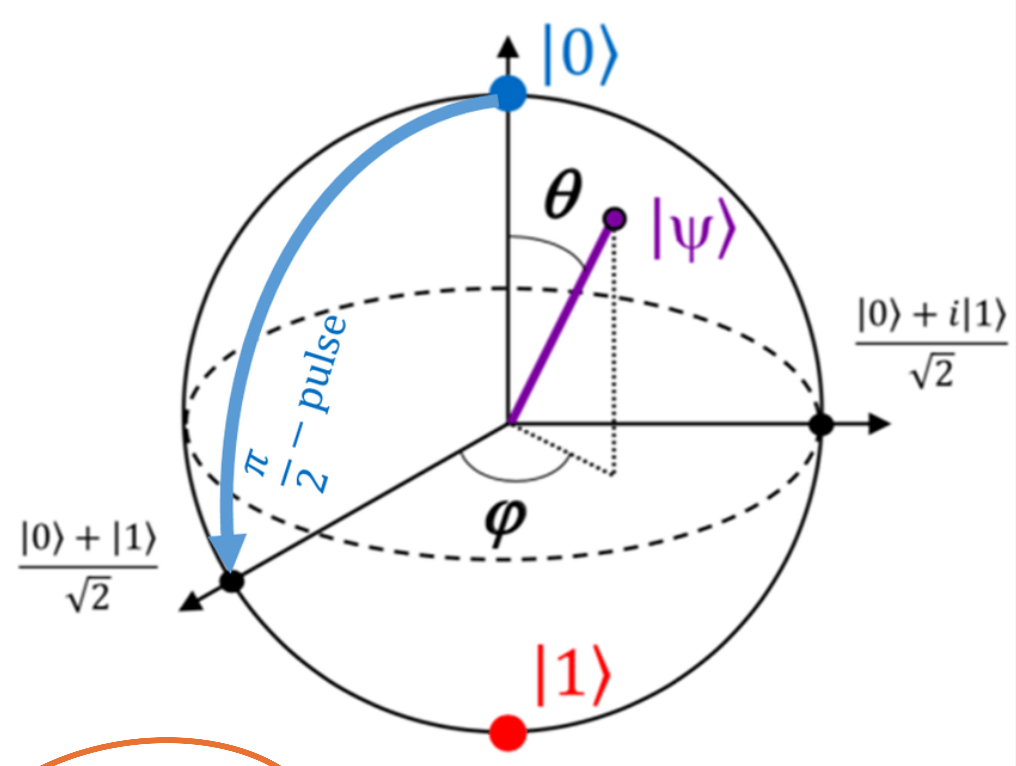
- Classical information processing uses a bit as a unit of information. A bit takes a value 0 or 1.

- n bits take one of 2^n different values from 000…0 to 111…1.
- A qubit is a unit of quantum information. It is a vector living in $\mathbb{C}^2$ .Basis vectors of $\mathbb{C}^2$ are $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$.
- Thus, $|\psi\rangle = \begin{pmatrix}\alpha\\\beta\end{pmatrix} = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix}$,where $|\alpha|^2$ and $|\beta|^2$ are the probabilities to measure $|0\rangle$ and $|1\rangle$,respectively (therefore, $|\psi\rangle$ is normalized: $|\alpha|^2 + |\beta|^2 = 1$.)

- $|\ \rangle$ (ket) is a symbol for a column vector. With a general qubit state represented as $|\psi\rangle = a_0|0\rangle + a_1|1\rangle, a_0, a_1 \in \mathbb{C}$ .0 $\leftrightarrow$ $|0\rangle$,1 $\leftrightarrow$ $|1\rangle$. In this sense, $|\psi\rangle$ is 0 and 1 simultaneously.

When we write these vectors in polar coordinates, we get the Bloch sphere

- Corresponding to $|\psi\rangle$, there is a “dual” vector $\langle\psi| = (\bar{a}_0, \bar{a}_1)$.
- We use the convention that $|\psi\rangle$ is normalized; $\sqrt{\langle\psi|\psi\rangle} = \sqrt{|a_0|^2 + |a_1|^2} = 1$.
- The reason for this becomes clear later.



The coefficient of $|0\rangle$ may be taken real.

$$|\psi\rangle = cos\frac{\theta}{2}|0\rangle + e^{i\varphi}sin\frac{\theta}{2}|1\rangle$$

Source: The Bloch sphere provides a useful means of visualizing the state of a... | Download Scientific Diagram (researchgate.net)

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Exercise

- Let $|\psi\rangle = 2|0\rangle + i\,|1\rangle$.

(1) Normalize $|\psi\rangle$.

(2) What is the probability of measuring $|0\rangle$ when $|\psi\rangle$ is measured.

(3) What is the probability of measuring $|1\rangle$ when $|\psi\rangle$ is measured.

Answer

(1) $\langle\psi|\psi\rangle = 5; |\psi\rangle = \frac{1}{\sqrt{5}}(2|0\rangle + i|1\rangle)$.

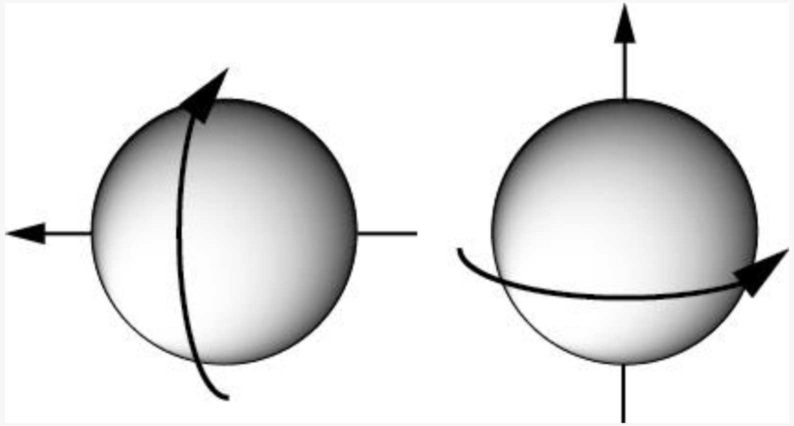
(2) $P(0) = \langle\psi|0\rangle\langle 0|\psi\rangle = \frac{4}{5}$.

(3) $P(1) = \langle\psi|1\rangle\langle 1|\psi\rangle = \frac{1}{5}$.

Multi-qubit Systems and Entanglement

- Consider a system made of 2 qubits. If the state of the first qubit is $|0\rangle$ and the second qubit is $|1\rangle$, the total state is written as $|0\rangle|1\rangle$ or $|01\rangle$ or $|0\rangle \otimes |1\rangle$.
- A state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|0\rangle)$ is decomposed as $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle$. It is a tensor product state or a product state.

$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

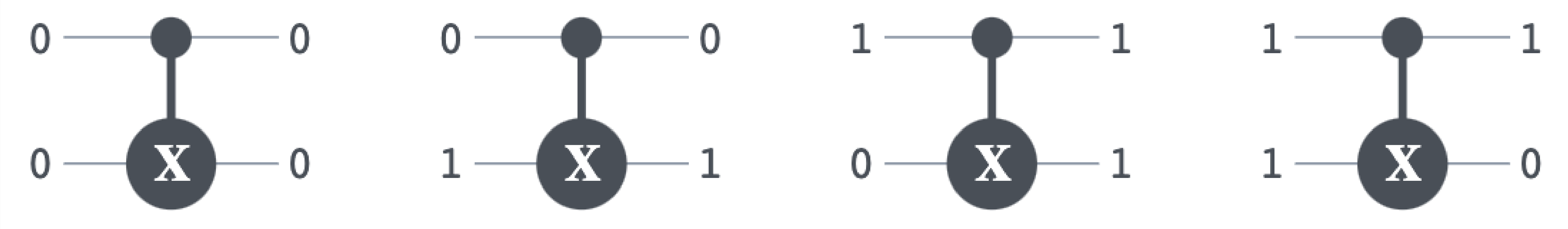


$|0\rangle$

- Measure the first qubit → $50\%\ |0\rangle,\ \ 50\%\ |1\rangle$. Measure the second qubit → $100\%\ |0\rangle$ independently of the outcome of the first qubit. They are not correlated.

Activity 2: A 2-qubit gate

- With two qubits, the gate flips the second (target qubit) if the first (control qubit) is in state $|1\rangle$
- If the control qubit is in the $|0\rangle$ state, nothing happens to the target qubit..



- This gate is called **CNOT** (short for Controlled NOT)

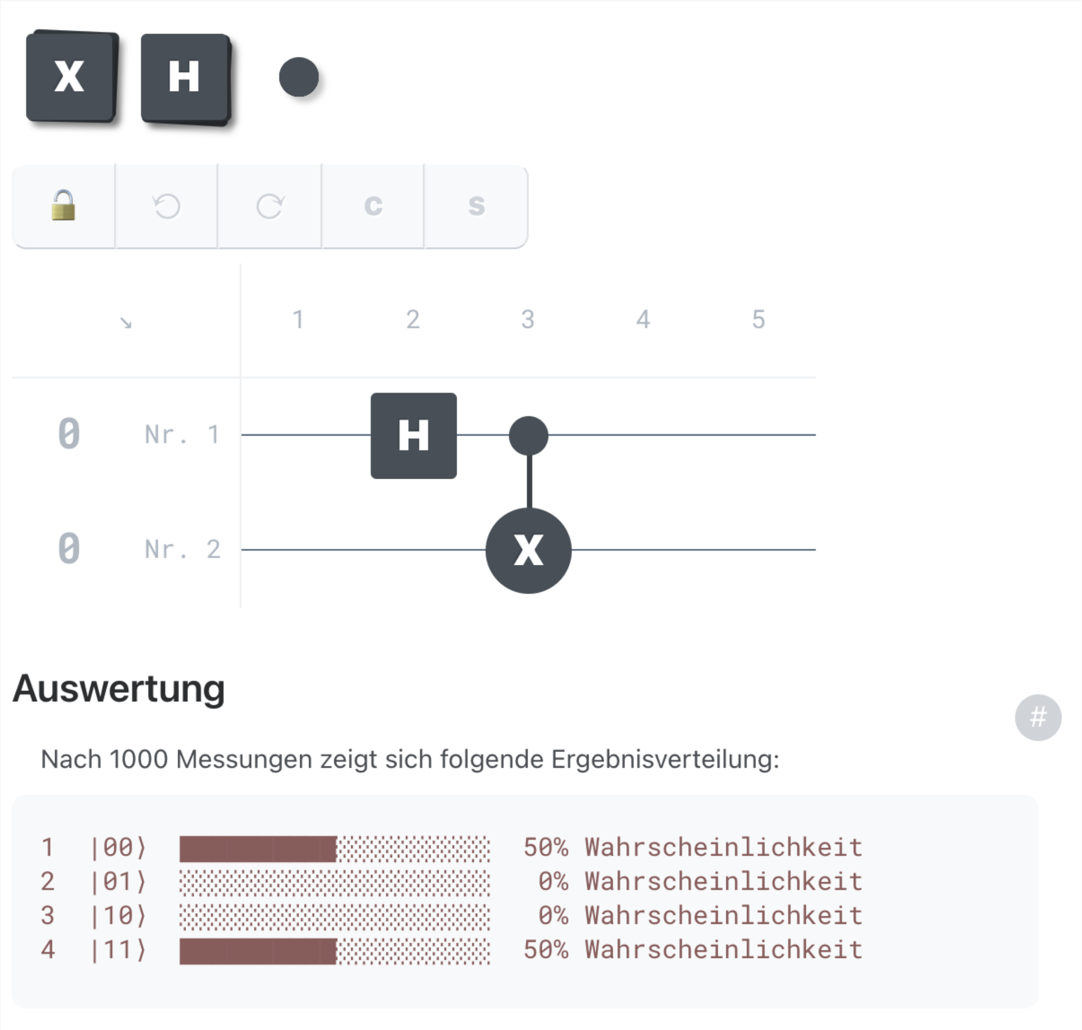
Activity 2: What happens, if the control qubit is in a superposition state of both $|0\rangle$ and $|1\rangle$?

Open https://bit.ly/iqm-2 to check it out!

Activity 2 - Discussion

Hadamard and CNOT

- If the **control qubit** is in a superposition, the measurement result of the target qubit depends on which value the control qubit now assumes.
- This combination of gates "**entangles**" the two qubits.
- If the state of one of the qubits is then measured, the state of the other qubit is automatically known.
- Viewed as a whole, the two qubits have a well-defined state, but the individual qubits cannot be assigned their own well-defined state.
- Only with entanglement is it possible to create truly arbitrary states



Activity 2: Description of a two-qubit quantum state

A two-qubit quantum state $|\psi\rangle$ can be described mathematically as

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

with $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$

The entangled state we just created can be written as

$$|\psi\rangle = {}^{1}/_{\sqrt{2}}|00\rangle + {}^{1}/_{\sqrt{2}}|11\rangle$$

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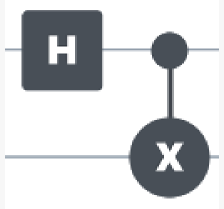
Entanglement

"Viewed as a whole, the two qubits have a well-defined state, but the individual qubits cannot be assigned their own well-defined state."



The state we generated was: $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

Let's assume, it is comprised of two independent qubits, meaning we could write:

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \stackrel{?}{=} (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$$

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \stackrel{?}{=} ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$$

Symbol for tensor product

Comparing both sides yields to:

$$\frac{1}{\sqrt{2}} = ac, 0 = ad, 0 = bc, \frac{1}{\sqrt{2}} = bd$$

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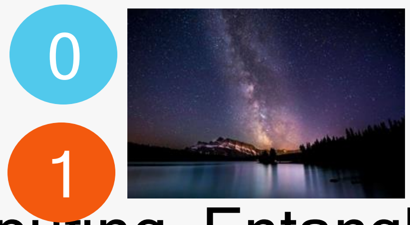
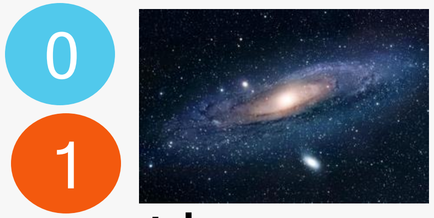
Multi-qubit Systems and Entanglement

- We have seen, the two qubits are entangled! If the1st qubit is measured 0, the 2nd qubit is always measured 0. Two outcomes are correlated. This is true no matter how far the two qubits are separated. Faster than the speed of light propagation? No, nothing propagates.





- Entanglement plays an essential role in quantum computing. Entanglement is a purely quantum correlation and quantum computing takes advantage of this.

- A two-qubit state is a vector $\in \mathbb{C}^4$. It is expanded in terms of $|00\rangle, |01\rangle, |10\rangle, |11\rangle$; $|\Psi\rangle = \sum_{i_k=0}^{1} c_{i_1 i_0} |i_1 i_0\rangle = \sum_{x=0}^{3} c_x |x\rangle$. $x \in \{0,1,2,3\}$ is the decimal notation of $(i_1 i_0)_2$. $c_x \in \mathbb{C}, \sum_x |c_x|^2 = 1$.

Exercise

- Which of the following states are product states? Decompose product states into component states and find the probability distributions of the measurement outcomes of these states.

$$(1)\ \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$(2)\ \frac{1}{\sqrt{2}}(|01\rangle - |00\rangle)$$

$$(3)\ \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Activity 3 – More qubits, more fun

Open https://bit.ly/iqm-3 and create a circuit that entangles all five qubits.

Take it further:

- Remove all the gates (or reload the page) and check out the effects of a T gate. Use it without any other gate and then sandwich it between two Hadamard gates.

Multi-qubit Systems and Entanglement

- Two-qubit states cannot be visualized by two vectors on the Bloch sphere in general. It is possible only for product states. Recall that an entangled state cannot be factored into a product of two one-qubit states.
- n-qubit system has 2^n basis vectors $\{|00 \ldots 00\rangle, |00 \ldots 01\rangle, \ldots, |11 \ldots 11\rangle\}$ or $\{|0\rangle, |1\rangle, \ldots, |2^n - 1\rangle\}$ in decimal notation. $|\Psi\rangle = \sum_{x=0}^{2^n-1} c_x |x\rangle$ is a superposition of 2^n basis vectors.

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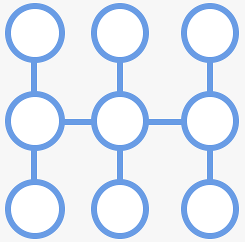
Quantum Computing Eras

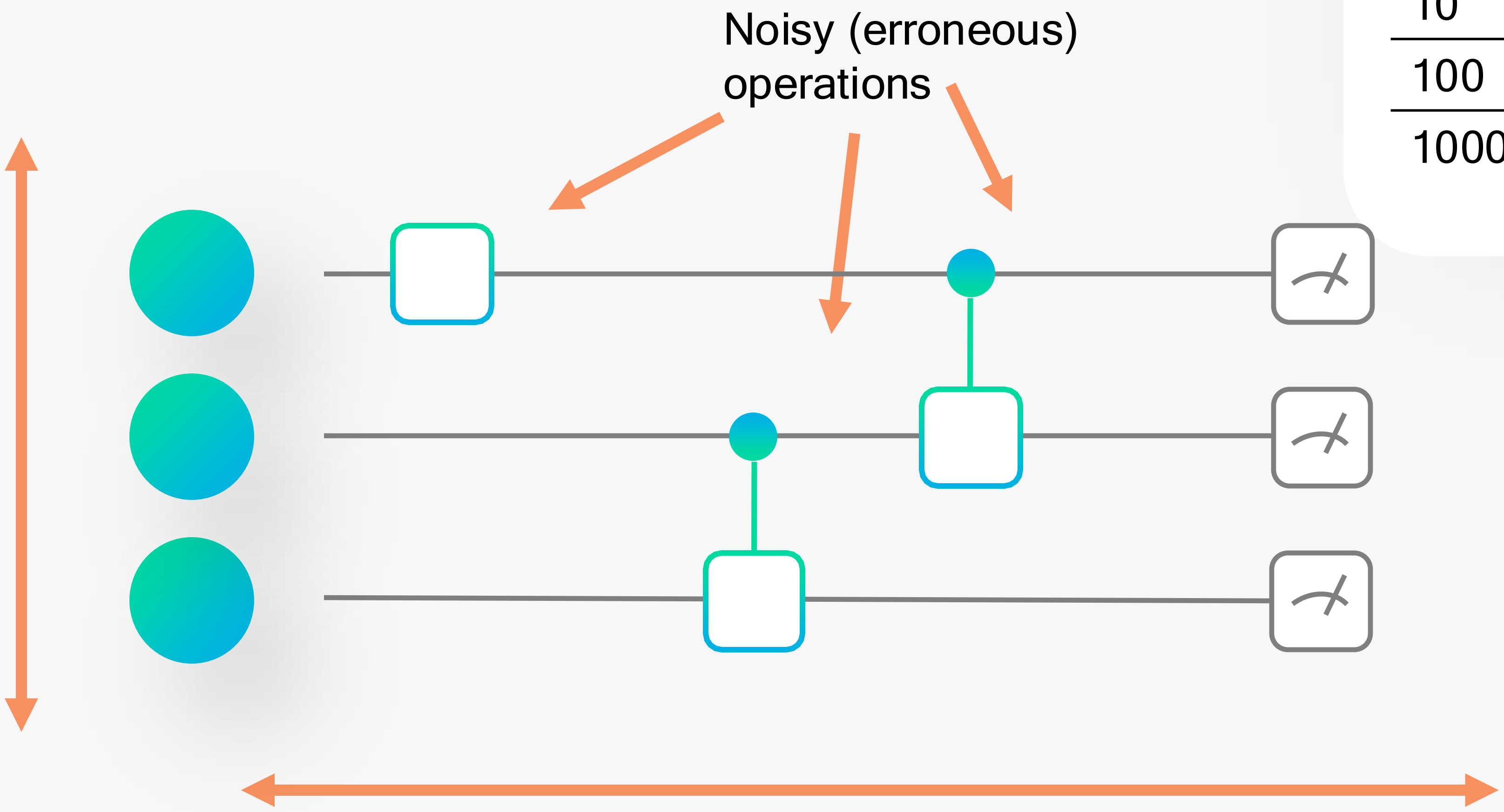
Quantum computers: NISQ era

# gates in algorithm	Algorithm fidelity
1	99.9%
10	99.0%
100	90.5%
1000	36.8%

Error rate: 1/1000 = 0.1%

Moderate number of qubits with limited connectivity *(intermediate scale)*



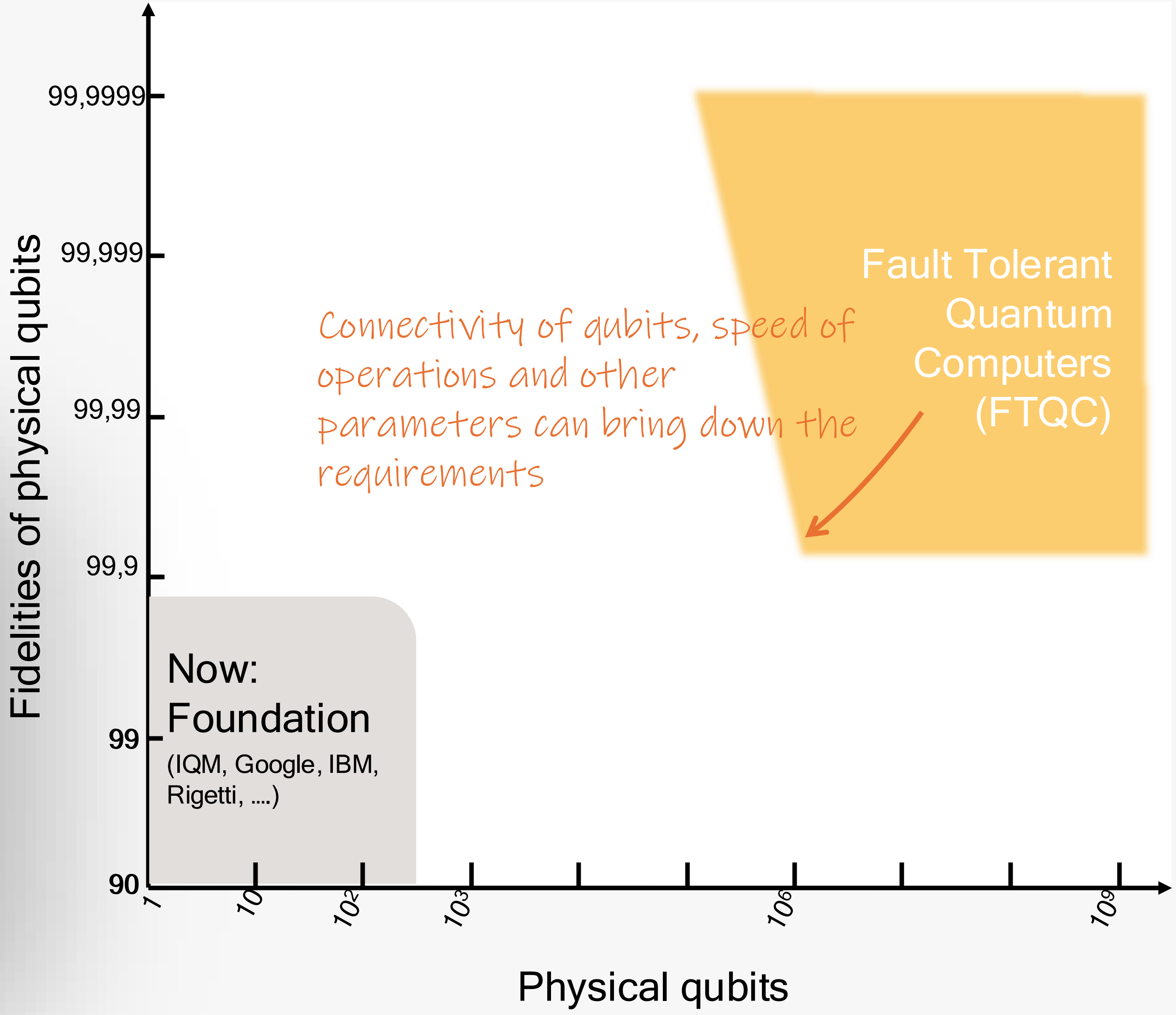


NISQ and FTQC

- Horizon: Fault Tolerant (FT) area

Why so complex?

- Error correction is based on redundancy
- 1 single fault tolerant qubit requires multiple physical qubits
- Noise needs to be reduced below a certain threshold



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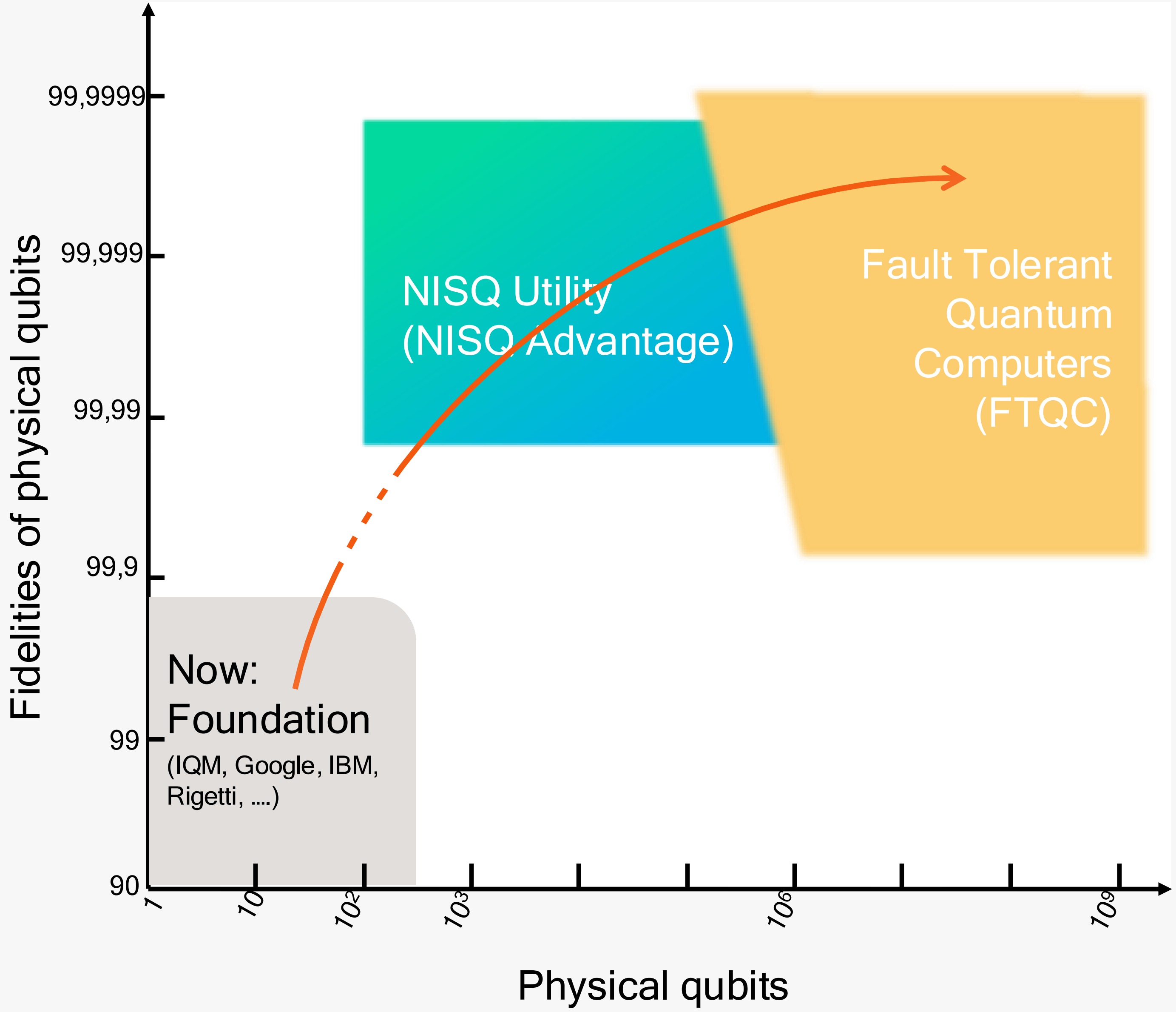
NISQ and FTQC

- Horizon: Fault Tolerant (FT) area
- But before that: NISQ Utility (NISQ Advantage)
  - Hybrid approaches
  - Potential for NISQ quantum utility
  - Solving tasks
    - faster,
    - better,
    - or using less energy



Conclusion

Conclusion: Foundations

- Qubits can not only be in the two states $|0\rangle$ and $|1\rangle$, but also in a **superposition** of $|0\rangle$ and $|1\rangle$. In this case, they have a certain probability of being measured as $|0\rangle$ or $|1\rangle$. However, a measurement destroys the superposition.
- Quantum computers use special quantum gates to manipulate the state of qubits.
- Even though the measurement results of a qubit are random, the qubit is still always in a precisely defined state.
- Two qubits can **be made to interact** with each other. If you then measure the state of one of the qubits, you automatically know the state of the other qubit.